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Derivation of Certain Relations Involving Sums of Determinants

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IN THIS paper we prove certain theorems on sums of determinants by means of partial derivative operators and generalized Kronecker symbols. Other methods of approach have been, for the most part, rather long and difficult. Several known theorems have been generalized and a few new theorems discovered and proved.

In Part I we employ partial derivative operators in a manner as first suggested in a paper by Stouffer[1] in 1925.

In Part II we employ the generalized Kronecker symbol as introduced by Muraghan[2] in the same year.

I

The expansion of a determinant $A = |a_{rs}|$ of order n with respect to row r takes the form

$$A = a_{r1} A_{r1} + a_{r2} A_{r2} + \ldots + a_{rn} A_{rn}$$

where A_{rs} is the cofactor of element a_{rs}. Unless otherwise specified it will be assumed that the n^2 elements constituting any determinant are mutually independent. It follows that

$$\frac{\partial}{\partial a_{rs}} A = A_{rs}, \text{ while } \sum_{s=1}^{n} a_{rs} \frac{\partial}{\partial a_{rs}} A = A.$$

Again it is evident that the operator $\sum_{s=1}^{n} b_{rs} \frac{\partial}{\partial a_{rs}}$ acting on A will replace the elements of row r by the elements $b_{r1}, b_{r2}, \ldots, b_{rn}$; the operator $\left(1 - a_{rs} \frac{\partial}{\partial a_{rs}}\right)$ acting on A will replace the element a_{rs} by a zero element; and the operator $\left(1 - 2\, a_{rs} \frac{\partial}{\partial a_{rs}}\right)$ acting on A will change the sign of the element in row r and column s.

To avoid the printing of a subscript to an index character, and to condense the notation we shall indicate the element in row r_1 and column s_1 by $a_{rs(1)}$; the partial derivative $\partial/\partial a_{rs(1)}$ by $E_{rs(1)}$; and the repeated operator $(E_{rs(k)}) \ . \ . \ . \ (E_{rs(2)})(E_{rs(1)})$ by $\overset{k}{\underset{m=1}{R}} E_{rs(m)}$.

If M is the minor of order k of the determinant A, in which the rows $r_1, r_2, \ldots . r_k$ and columns $s_1, s_2, \ldots . s_k$ are represented, then the algebraic complement of M will be denoted by $A_{rs(1,\ 2,\ \ldots .,\ k)}$. If, however, the rows and columns are definitely prescribed, as for instance rows 2, 5, 6, 8, columns 1, 3, 4, 8, then we shall represent the corresponding algebraic complement by the usual notation $A_{2568,\ 1348}$.

THEOREM I. Let $A = |a_{rs}|$ be a determinant of order n and D be a determinant of the same order formed from A by substituting zero elements for the k elements $a_{rs(1)}, \cdots, a_{rs(k)}$, no two of which are in the same row or column; then

$$D = A - \Sigma\, a_{rs(1)} A_{rs(1)} + \Sigma\, a_{rs(1)}\, a_{rs(2)} A_{rs(1,\ 2)}$$
$$- \ . \ . \ . \ . \ . \ .$$
$$+ (-1)^k a_{rs(1)} \cdots a_{rs(k)} A_{rs(1,\ 2,\ \ldots .\ k)} \qquad (1)$$

where $\Sigma\, a_{rs(1)}\, a_{rs(2)} \cdots a_{rs(m)} A_{rs(1,\ 2,\ \ldots .,\ m)}$ represents the sum of $\binom{k}{m}$ terms, each of which is formed by taking the product of m of the k deleted elements into the algebraic complement of the minor in which the m rows and m columns containing these m elements are represented.

By hypothesis the k indices $r_1, \ldots . r_k$ are all distinct, the indices $s_1, \ldots . s_k$ are all distinct; and $k \leqq n$.

The operator $(1 - a_{rs(1)} E_{rs(1)})$ when applied to A replaces the element $a_{rs(1)}$ by zero, and the repeated operator $\overset{k}{\underset{m=1}{R}} (1 - a_{rs(m)} E_{rs(m)})$ when applied to A replaces the elements $a_{rs(2)}, \cdots, a_{rs(k)}$ by zeros. Therefore

$$D = [\overset{k}{\underset{m=1}{R}} (1 - a_{rs(m)} E_{rs(m)})]A. \qquad (2)$$

Upon expanding the right-hand member of (2) the equality takes the form

$$D = [1 - \Sigma\, a_{rs(1)} E_{rs(1)} + \Sigma\, a_{rs(1)}\, a_{rs(2)} E_{rs(1)} E_{rs(2)}$$
$$- \ . \ . \ . \ . \ . \ .$$
$$+ (-1)^k a_{sr(1)} \cdots a_{rs(k)} E_{rs(1)} \cdots E_{rs(k)}]\, A \qquad (3)$$

which is equivalent to the equality (1) given in the theorem.

If the k elements are chosen as the first k elements of the main diagonal, this theorem becomes the following corollary proved by Stetson[3] in 1904.

COROLLARY I. If D is the determinant of order n formed by substituting zero elements for the first k elements of the main diagonal in A, then

$$D = A - \Sigma a_{11} A_{11} + \Sigma a_{11} a_{22} A_{12,12} - \cdots + (-1)^k a_{11} \cdots a_{kk} A_{1 \cdots k,\, 1 \cdots k}. \quad (4)$$

If in addition to the above conditions we assume $k = n$, a second form results which is identical with the original theorem on this type of expansion proved by Muir[4] in 1898.

COROLLARY II. If D is the invertebrate determinant of order n formed by substituting zero elements for the main diagonal elements in A, then

$$D = A - \Sigma a_{11} A_{11} + \Sigma a_{11} a_{22} A_{12,12} - \cdots + (-1)^k a_{11} a_{22} \cdots a_{nn}. \quad (5)$$

THEOREM II. Let determinants A and D be as defined in Theorem I; then

$$D = A - a_{rs(1)} |A_{rs(1)}|_0 - a_{rs(2)} |A_{rs(2)}|_0 - \cdots - a_{rs(k)} |A_{rs(k)}|_0, \quad (6)$$

where $|A_{rs(m)}|_0$ represents the determinant formed from $A_{rs(m)}$ by substituting zeros for the elements $a_{rs(m+1)}, \cdots, a_{rs(k)}$.

The right-hand member of equation (2) may be written as $(1 - a_{rs(1)} E_{rs(1)})[\overset{k}{\underset{i=2}{R}} (1 - a_{rs(i)} E_{rs(i)}) A]$, since the elements of A are mutually independent. If we now add $a_{rs(1)} E_{rs(1)}[\overset{k}{\underset{i=2}{R}} (1 - a_{rs(i)} E_{rs(i)}) A]$ the result is $[\overset{k}{\underset{i=2}{R}} (1 - a_{rs(i)} E_{rs(i)}) A]$. From $k - 1$ similar additions will result $(1 - a_{rs(k)} E_{rs(k)}) A$ which, if the term $a_{rs(k)} E_{rs(k)} A$ be added, will equal A. That is,

$$[\overset{k}{\underset{i=1}{R}} (1 - a_{rs(i)} E_{rs(i)})] A + \overset{k-1}{\underset{m=1}{\Sigma}} [a_{rs(m)} E_{rs(m)} \overset{k}{\underset{i=m+1}{R}} (1 - a_{rs(i)} E_{rs(i)})] A + a_{rs(k)} E_{rs(k)} A = A \quad (7)$$

Upon substituting D for the first term of the equality and transposing the rest of the left-hand member we have

$$D = A - \sum_{m=1}^{k-1} [a_{rs(m)} E_{rs(m)} \overset{k}{R}_{i=m+1} (1 - a_{rs(i)} E_{rs(i)})] A$$
$$- a_{rs(k)} E_{rs(k)} A, \qquad (8)$$

which equation is equivalent to (6).

We can at once state two important corollaries of this theorem.

COROLLARY I. If D be the determinant of order n formed from A by substituting zero elements for the first k elements of the main diagonal, then

$$D = A - a_{11}|A_{11}|_0 - a_{22}|A_{22}|_0 - \cdots - a_{kk} A_{kk}. \qquad (9)$$

COROLLARY II. If D be the determinant of order n formed from A by substituting zero elements for the n elements of the main diagonal, then

$$D = A - a_{11}|A_{11}|_0 - a_{22}|A_{22}|_0 - \cdots - a_{nn} A_{nn}. \qquad (10)$$

This last corollary was stated as a theorem by Horta[5] in 1890.

THEOREM III. Let $A = |a_{rs}|$ be a determinant of order n and Δ be a determinant formed from A by deleting k elements $a_{rs(1)}, \cdots, a_{rs(k)}$, no two of which are in the same row or column, and substituting respectively the elements $b_{rs(1)}, \cdots, b_{rs(k)}$ where $b_{rs} = a_{rs} - x_{rs}$; then

$$A = \Delta + \sum x_{rs(1)} \Delta_{rs(1)} + \sum x_{rs(1)} x_{rs(2)} \Delta_{rs(1,2)} + \cdots$$
$$+ x_{rs(1)} \cdots x_{rs(k)} \Delta_{rs(1,2,\cdots,k)}. \qquad (11)$$

where $\sum x_{rs(1)} \cdots x_{rs(m)} \Delta_{rs(1,2\cdots,k)}$ represents the sum of $\binom{k}{m}$ term each consisting of the product of m of the k elements $x_{rs(1)}, \cdots, x_{rs(k)}$, into the algebraic complement of the minor in which the rows $r_1, \cdots, r_m$ and columns $s_1, \cdots, s_m$ are represented. The elements x_{rs} are any k variables which are mutually independent of the elements a_{rs}.

The single operator $(1 + x_{rs(1)} E_{rs(1)})$ applied to Δ will substitute the element $a_{rs(1)}$ for $b_{rs(1)}$; and the repeated operator $\overset{k}{R}_{m=1} (1 + x_{rs(m)} \Sigma_{rs(m)})$ applied to Δ will substitute "a" elements for the respective "b" elements hence:

$$A = [\overset{k}{R}_{m=1} (1 + x_{rs(m)} E_{rs(m)})] \Delta. \qquad (12)$$

Upon expanding, the right-hand member of (12) takes the form

$$[1 + \Sigma\, x_{rs(1)}\, E_{rs(1)} + \Sigma\, x_{rs(1)}\, x_{rs(2)}\, E_{rs(1)}\, E_{rs(2)} + \cdots + x_{rs(1)} \cdots x_{rs(k)}\, E_{rs(1)} \cdots E_{rs(k)}]\, \Delta \qquad (13)$$

which is identically the right-hand member of the equality (11).

THEOREM IV. If A and D are defined as in the preceding theorems, then

$$A = D + \Sigma\, a_{rs(1)}\, D_{rs(1)} + \Sigma\, a_{rs(1)}\, a_{rs(2)}\, D_{rs(1,2)} + \cdots + a_{rs(1)} \cdots a_{rs(k)}\, D_{rs(1,2,\cdots,k)} \qquad (14)$$

This theorem follows at once if we consider the limiting form of Theorem III when the variables $x_{rs(m)}$ approach as limits the respective elements $a_{rs(m)}$.

Stetson[3], in proving a generalization of Muir's Expansion Theorem (Th. 1., Cor. I.), obtained zero elements in his determinants by employing a compound element similar to b_{rs}.

COROLLARY I. If D be the determinant of order n formed from A by substituting zero elements for the first k elements of the main diagonal, then

$$A = D + \Sigma\, a_{11}\, D_{11} + \Sigma\, a_{11}\, a_{22}\, D_{12,12} + \cdots + a_{11} \cdots a_{kk}\, A_{12\cdots k,12\cdots k}. \qquad (15)$$

COROLLARY II. If D be the determinant of order n formed from A by substituting zero elements for the n elements of the main diagonal, then

$$A = D + \Sigma\, a_{11}\, D_{11} + \Sigma\, a_{11}\, a_{22}\, D_{12,12} + \cdots + a_{11} \cdots a_{nn}. \qquad (16)$$

Corollary II was first stated as a theorem by Cayley[6] in 1848 and is known as Cayley's Expansion Theorem. Corollary I, which is more general than Corollary II, was first stated as a theorem by Muir[7] in 1896.

THEOREM V. Let $A = |a_{rs}|$ be a determinant of order n and δ be a determinant of the same order formed from A by changing the signs of the k elements $a_{rs(1)}, a_{rs(2)}, \cdots, a_{rs(k)}$, no two of which are in the same row or column; then

$$\delta = A - 2\Sigma\, a_{rs(1)}\, A_{rs(1)} + 2^2\Sigma\, a_{rs(1)}\, a_{rs(2)}\, A_{rs(1,2)} - \cdots + (-2)^k\, a_{rs(1)} \cdots a_{rs(k)}\, A_{rs(1,2,\cdots,k)} \qquad (17)$$

where the terms in the expansion are as defined in the first theorem.

The single operator $(1-2\,a_{rs(1)}\,E_{rs(1)})$ when applied to A changes the sign of $a_{rs(1)}$; and the repeated operator $\overset{k}{\underset{m=1}{R}}\,(1-2\,a_{rs(m)}\,E_{rs(m)})$ when applied to A changes the signs of the k elements $a_{rs(1)}, a_{rs(2)}, \cdots, a_{rs(k)}$.

Hence

$$\delta = [\overset{k}{\underset{m=1}{R}}\,(1-2\,a_{rs(m)}\,E_{rs(m)})]A. \tag{18}$$

Upon expanding the right-hand member of (18) the equality takes the form

$$\begin{aligned}\delta = [1 &- 2\,\Sigma\, a_{rs(1)}\,E_{rs(1)} + 2^2\,\Sigma\, a_{rs(1)}\,a_{rs(2)}\,E_{rs(1)}\,E_{rs(2)} \\ &- \cdots \\ &+ (-2)^k\, a_{rs(1)}\,a_{rs(2)}\cdots a_{rs(k)}\,E_{rs(1)}\,E_{rs(2)}\cdots E_{rs(k)}]\,A\end{aligned} \tag{19}$$

which is equivalent to the equality of (17).

II.

The ordinary Kronecker symbol is denoted by δ_r^s and has the value zero or unity according as r and s are different or identical. The two indices r and s assume independently any integral values within the range $1, 2, \cdots n$.

The general Kronecker symbol as introduced by Murnaghan[2] may be denoted by

$$\begin{bmatrix} s_1 s_2 \cdots s_m \\ r_1 r_2 \cdots r_m \end{bmatrix}$$

where the indices r_i, s_i assume independently any integral values within the range $1, 2, \cdots n$; and $m \leqq n$.

This generalized symbol is characterized by the following postulates or definitions:

(1) If the elements r_i are not all distinct, or if the elements s_i are not all distinct, the generalized symbol is taken to be zero.

(2) If, neglecting order, the set of elements r_i is in any way different from the set of elements s_i then the generalized symbol is again taken to be zero.

(3) If the elements r_i are all distinct, and if the set of elements s_i is identical with the set of elements r_i, then the symbol is taken to be plus one or minus according as the two sets become identical in order after an even or an odd number of inversions within either set.

We shall adopt the common convention that whenever a Greek

letter index appears twice in a term that term is to be summed for all values of the index within the range $1, 2, \cdots n$.

The determinant A may now be defined in several equivalent forms:

$$A \equiv \begin{bmatrix} \sigma_1 \sigma_2 \cdots \sigma_n \\ r_1 r_2 \cdots r_n \end{bmatrix} a_{r\sigma(1)} a_{r\sigma(2)} \cdots a_{r\sigma(n)}$$

$$\equiv \begin{bmatrix} s_1 s_2 \cdots s_n \\ \rho_1 \rho_2 \cdots \rho_n \end{bmatrix} a_{\rho s(1)} a_{\rho s(2)} \cdots a_{\rho s(n)} .$$

$$\equiv \frac{1}{n!} \begin{bmatrix} \sigma_1 \sigma_2 \cdots \sigma_n \\ \rho_1 \rho_2 \cdots \rho_n \end{bmatrix} a_{\rho\sigma(1)} a_{\rho\sigma(2)} \cdots a_{\rho\sigma(n)} .$$

The algebraic complement $A_{r\sigma(1, 2, \cdots k)}$ is identically

$$\begin{bmatrix} \sigma_1 \sigma_2 \cdots \sigma_n \\ r_1 r_2 \cdots r_n \end{bmatrix} a_{r\sigma(k+1)} \cdots a_{r\sigma(n)} .$$

THEOREM VI. If A is a determinant of order n and $B, C, \cdots, L$ are p other determinants of the same order, the sum of the $(n)(n-1) \cdots (n-p+1)$ determinants composed of $n-p$ rows from A and one row from each of the p determinants $B, C, \cdots, L$ is equal to that of the same number of determinants composed of $n-p$ columns from A and one column from each of the p determinants $B, C, \cdots, L$. The columns and rows are to occupy the same places in the new determinants as in the originals, and $p \leqq n-1$.

First we recognize

$$\begin{bmatrix} \sigma_1 \sigma_2 \cdots \sigma_n \\ r_1 r_2 \cdots r_n \end{bmatrix} b_{r\sigma()1} c_{r\sigma(2)} \cdots l_{r\sigma(p)} a_{r\sigma(p+1)} \cdots a_{r\sigma(n)}$$

to be a determinant composed of $n-p$ rows from A and one row from each of the p determinants $B, C, \cdots, L$. Furthermore

$$\frac{1}{(n-p)!} \begin{bmatrix} \sigma_1 \sigma_2 \cdots \sigma_n \\ \rho_1 \rho_2 \cdots \rho_n \end{bmatrix} b_{\rho\sigma(1)} c_{\rho\sigma(2)} \cdots l_{\rho\sigma(p)} a_{\rho\sigma(p+1)} \cdots a_{\rho\sigma(n)} \tag{20}$$

represents the sum of the $n(n-1) \cdots (n-p+1)$ determinants composed of $n-p$ rows from A and one row from each of the p determinants $B, C, \cdots, L$.

Again we recognize

$$\begin{bmatrix} s_1 s_2 \cdots s_n \\ \rho_1 \rho_2 \cdots \rho_n \end{bmatrix} b_{\rho s(1)} c_{\rho s(2)} \cdots l_{\rho s(p)} a_{\rho s(p+1)} \cdots a_{\rho s(n)}$$

to be a determinant composed of $n-p$ columns from A and one column from each of the p determinants. Furthermore

$$\frac{1}{(n-p)!}\left[\begin{matrix}\sigma_1\,\sigma_2\cdot\cdot\cdot\sigma_n\\ \rho_1\,\rho_2\cdot\cdot\cdot\rho_n\end{matrix}\right] b_{\rho\sigma(1)}\, c_{\rho\sigma(2)}\cdot\cdot\cdot l_{\rho\sigma(p)}\, a_{\rho\sigma(p+1)}\cdot\cdot\cdot a_{\rho\sigma(n)} \tag{21}$$

represents the sum of the $n(n-1)\cdot\cdot\cdot(n-p+1)$ determinants composed of $n-p$ columns from A and one column from each of the p determinants. It is identical with (20).

An interesting example of the theorem occurs when $p+1=n$, in which case one row is taken from each of the n given determinants $A, B, C, \cdot\cdot\cdot, L$ for the construction of each new one.

Another example of interest is obtained if $p=1$, in which instance one row is taken from B and the remaining rows from A in the construction of each new determinant. This example was stated as a theorem by Muir[8] in 1888.

If $B=C=D=\cdot\cdot\cdot=L$ the theorem reduces at once to:

COROLLARY I. If A and B are two determinants of order n, the sum of the $\binom{n}{p}$ determinants of the same order composed of $n-p$ rows from A and of p rows from B is equal to that of the $\binom{n}{p}$ determinants composed of $n-p$ columns from A and of p columns from B.

This corollary was given as a lemma by J. Deruyts[9] in 1883.

A second corollary is obtained if certain groups of the determinants are identical. Suppose that k_1 determinants are identical with B, k_2 determinants are identical with C, and so on, where $\Sigma k_i \leqq n$.

COROLLARY II. If $A, B, C, \cdot\cdot\cdot, L$ are $p+1$ determinants of order n the sum of the $\dfrac{n(n-1)\cdot\cdot\cdot(n-p+1)}{(k_1!)(k_2!)\cdot\cdot\cdot(k_p!)}$ determinants each composed of k_1 rows from B, k_2 rows from $C, \cdot\cdot\cdot, k_p$ rows from L, and the remaining $n-\Sigma k_i$ rows from A, is equal to the sum of the same number of determinants each composed of k_1 columns from B, k_2 columns from $C, \cdot\cdot\cdot, k_p$ columns from L, and the remaining $n-\Sigma k_i$ columns from A. The columns and rows are to occupy the same places in the new determinants as in the originals and $\Sigma k_i \leqq n$.

THEOREM VII. If $\delta_{k_i}=|a_{rh}|$, $r=1,2,\cdot\cdot\cdot n$, $h=k_1, k_1+1, \cdot\cdot\cdot, n, n+1, 1, 2, \cdot\cdot\cdot, k_1-2$, and if $k_1, k_2, \cdot\cdot\cdot, k_p$ are p distinct positive integers each different from unity and less than $n+1$; then the sum of the $p!\,\binom{n}{p}$ determinants of order n, composed of

$n-p$ rows from $A=|a_{rs}|$ and one row from each of the p determinants $\delta_k, \delta_k, \cdots, \delta_k$ is equal to $(-1)^{\Sigma k} p!\, \Delta_{n+p+1-\Sigma k}$ where Δ_r denotes the determinant obtained by deleting the r the column of the n by $n+1$ array $||a_{rs}||$ $r=1,2,\cdots n$, $s=1,2,\cdots n+1$. If the number $n+1+p-\Sigma k$ is not between 1 and $n+1$, inclusive, then a convenient multiple of $n+1$ is to be added to it.

By Theorem VI the above sum of $p!\binom{n}{p}$ determinants is equal to the sum of the same number of determinants composed of $n-p$ columns from A and one column from each of the p determinants δ_{k_i}. Most of these determinants are zero due to duplicate columns.

In the determinant δ_{k_i} the elements in column s_i have for second index s_i+k_i-1. If s_i+k_i-1 is not between 1 and $n+1$, inclusive, then $n+1$ is to be subtracted from it. Now s_i+k_i-1 cannot equal s_i since by hypothesis $k_i \neq 1$. Hence to construct a nonvanishing determinant composed of $n-p$ columns from $A=|a_{rs}|$ and one column, with index s_i, from each of the p determinants δ_{k_i}, we must take $s_1+k_1-1=n+1$. Note again that the elements in column s_1 have then the second index equal to s_1+k_1-1. Having fixed s_1 we must next take $s_2+k_2-1=s_1=n+2-k_1$, $s_3+k_3-1=s_2=n+3-k_1-k_2$, etc.

For each arrangement of the p numbers k_i there is a corresponding set of indices s_i and one nonvanishing determinant. Since the p numbers k_i can be permuted in $p!$ ways there are just $p!$ nonvanishing determinants. All of these determinants are characterized by the absence of elements with the second index equal to

$$n+p+1-\Sigma k_i.$$

Hence each determinant is equal to $\Delta_{n+p+1-\Sigma k}$ except for the sign which is $(-1)^{\Sigma k}$.

If in the statement of the theorem we assume $k_1=k_2=\cdots=k_p$ a corollary follows at once.

COROLLARY I. The sum of the $\binom{n}{p}$ determinants of order n composed of $n-p$ from A and of p rows from δ_k is equal to $(-1)^{pk}\Delta_{n+p+1-pk}$.

This theorem[9] and corollary[10] were theorems of Deruyts published in 1883 and 1881. Le Paige[11] proved the corollary for the special case $p=1$, $k=2$ in 1880.

THEOREM VIII. If $A=|a_{rs}|$ is a determinant of order n, and if from A a new determinant be formed by altering the order of the

elements in the first row, a second determinant by making the same alteration in the elements of the second row, and so on, the sum of the n determinants thus derived is m times the original, where m is the number of elements of a row that after the change of order occupy the same positions as before.

To prove this theorem first construct a new determinant B from A by altering the order of the columns then apply Theorem VI. The sum of the n determinants each composed of one row from B and the remaining rows from A is equal to the sum of the n determinants each composed of one column from B and the remaining columns from A. But all the n determinants composed of one column from B and the remaining columns from A are zero due to duplicate columns except for those m determinants in which the column taken from B is identical with the one it replaces in A.

This theorem is equivalent to a theorem given by Deruyts[12] in 1885.

THEOREM IX. If $A = |a_{rs}|$ is a determinant of order n and

$$B = \begin{vmatrix} \lambda_1 a_{1n} & \lambda_2 a_{11} & \lambda_3 a_{12} & \cdots & \lambda_n a_{1\,n-1} \\ \lambda_1 a_{2n} & \lambda_2 a_{21} & \lambda_3 a_{22} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \lambda_1 a_{nn} & \lambda_2 a_{n1} & \lambda_3 a_{n3} & \cdots & \lambda_n a_{n\,n-1} \end{vmatrix}$$

where $\lambda_1, \lambda_2, \cdots \lambda_n$ are n arbitrary multipliers different from zero, then the sum of the $\binom{n}{p}$ determinants, each composed of p rows from B and $n - p$ rows from A, is equal to zero. The rows are to occupy the same positions in the new determinants as in the originals, and $p < n$.

Applying the statement of Theorem VI, Corollary I, it is evident that the determinants obtained by inserting columns from B into A are all zero due to duplicate columns.

This theorem was stated as a theorem for the special case of $p = 1, n = 4$ by Hammond[13] in 1881.

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